UDC 531.38

ON THE QUALITATIVE ANALYSIS OF MOTION OF A SOLID BODY IN THE GORIACHEV-CHAPLYGIN PROBLEM

PMM Vol. 41, № 2, 1977, pp. 225-233

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(Received June 17, 1976)

Qualitative properties of typical rotations of a heavy solid body are analized in the case of the Goriachev-Chaplygin problem in which the first integrals of equations of motion are independent. Gyration numbers of tangent vector fields on two-dimensional invariant tori are determined. It is shown that the nutation of a solid body is a quasi-periodic motion, and spin and precession have a principal motion. If the gyration number is irrational, then in the case of a solid body the principal motion of node lines is zero.

In the case of the Goriachev-Chaplygin problem of motion of a heavy solid body with a fixed point the principal moments of inertia satisfy the relation $A = B = 4C$ and the center of mass lies in the equatorial plane of the ellipsoid of inertia. Initial conditions are selected so that the constant of the area integral is zero. There exists then a particular supplementary integral, and this makes it possible to reduce the integration of the equations of motion to quadratures [1].

A qualitative investigation of the motion of the Goriachev-Chaplygin top was initiated by Sretenskii [2] who introduced in the equations of motion a small parameter related to the fast gyrations of the body, and outlined the pattern of motion in the first approximation with respect to that parameter. These investigations were continued in [3]. An analysis of the special variables introduced by Chaplygin for integrating the equations of motion is presented in [4]. The qualitative pattern of body gyration in some degenerate cases is investigated in [5].

Certain mathematical problems related to the analysis of the motion of a body in the Goriachev-Chaplygin problem are considered below without any simplifying assumptions.

1. Dynamic systems originating on invariant tori of the Goriachev-Chaplygin problem. For the symmetry of formulas we denote everywhere the Euler-Poisson variables $p, q, r, \gamma_1, \gamma_2$ and γ_3 by $x_1, x_2, \ldots, x_6$, respectively. In the Goriachev-Chaplygin problem the Euler-Poisson equations have four independent integrals

$$\begin{gathered}I_1 = 4(x_1^2 + x_2^2) + x_3^2 - 2\mu x_4, \quad I_2 = x_3(x_1^2 + x_2^2) + \mu x_1 x_6 \\ I_3 = 4(x_1 x_4 + x_2 x_5) + x_3 x_6 \quad (I_3 = 0), \quad I_4 = x_4^2 + x_5^2 + x_6^2 \\ (I_4 = 1)\end{gathered} \tag{1.1}$$

in which $\mu = Pr/C$, P is the body weight, r is the distance between the center of mass and the suspension point, and C is the moment of inertia about the dynamic symmetry axis.

We denote by $E(I_1, I_2)$ the common levels of the four integrals (1.1) in the six-dimensional space of the Euler-Poisson equations. Below we consider only such constants of integrals I_1 and I_2 for which functions (1.1) are independent at $E(I_1, I_2)$. The case in which $I_1 = 0$ is a specifically excluded. The remaining constants constitute a null set. If the integrals (1.1) are independent, E is a smooth two-dimensional manifold. A classical dynamic system obviously arises at E [6]. Such systems can be investigated by the method described in [7].

Each connected component of E is a two-dimensional torus [7]. The question arises of the number of connected components of E. A partial answer is provided by the following lemma.

Lemma 1. If μ is small, E is a union of two tori.

Proof. First, we assume $\mu = 0$. Then the common levels of functions I_1 and I_2 represent in the three-dimensional space R^3 $\{x_1x_2x_3\}$ two circles S_i^1 $(i = 1, 2)$ that lie in different planes $x_3 =$ const. To each point $\{x_1^\circ x_2^\circ x_3^\circ\}$ on S_i^1 $(i = 1,2)$ corresponds a circle cut out in the Poisson sphere $\{x_4^2 + x_5^2 + x_6^2 = 1\}$ by the area integral

$$4(x_1^\circ x_4 + x_2^\circ x_5) + x_3^\circ x_6 = 0$$

Since the position of that circle continuously depends on point $\{x_1^\circ x_2^\circ x_3^\circ\}$, the manifold E consists of two connected components when $\mu = 0$. If $\mu \neq 0$ but is small, then, by the Morse theorem the common levels are diffeomorphic to the level at $\mu = 0$ and, consequently, have as many connectedness components [8].

Note. If we increase μ, then, by the same Morse theorem, the number of connected components can change only then, when integrals (1.1) become interdependent.

On each two-dimensional invariant torus T^2 we can select angle variables φ_1 and φ_2 mod 2π in which the equations of motion are of the form

$$\varphi_1^{\cdot} = \omega_1, \quad \varphi_2^{\cdot} = \omega_2 \tag{1.2}$$

where ω_i $(i = 1, 2)$ are constants independent on I_1 and I_2. Equations (1.2) specify on T^2 a conditionally periodic motion at two frequencies ω_1 and ω_2. To determine these we introduce the variables s_1 and s_2 by formulas

$$x_3 = s_1 + s_2, \quad 4(x_1^2 + x_2^2) = -s_1 s_2$$

Note that variables s_1 and $-s_2$ were introduced by Chaplygin for integrating the equations of motion [1]. The Euler-Poisson variables can be expressed in terms of s_1 and s_2 with the use of integrals (1.1). In the new variables the equations of motion assume the form (cf. [1])

$$s_1^{\cdot} = \frac{\sqrt{\Phi(s_1)}}{2(s_1 - s_2)}, \quad s_2^{\cdot} = \frac{\sqrt{\Phi(s_2)}}{2(s_1 - s_2)} \tag{1.3}$$

$$(\Phi(z) = 4\mu^2 z^2 - (z^3 - I_1 z - 4I_2)^2)$$

Equations (1.3) are of the same form as in the Kowalewska problem [7]. Hence the results obtained there are valid for these equations. The variables s_1 and s_2 vary in the intervals $[a_1 b_1]$ and $[a_2 b_2]$, where the polynomial $\Phi(z) \geqslant 0$. If $I_2 \neq 0$, the intersection $[a_1 b_1] \cap [a_2 b_2]$ is empty. In the opposite case variables s_1 and s_2 may be the same, and since $s_1 s_2 \leqslant 0$, $s_1 = s_2 = 0$ on T^2 for certain initial data. Consequently, $x_1 = x_2 = x_3 = 0$ and $I_2 = 0$. A detailed qualitative analysis of the behavior of variables s_1 and s_2 is given in [4, 7].

The numbers a_i and b_i $(i = 1, 2)$ are simple roots of polynomial $\Phi(z)$, since otherwise asymptotic motions would exist on the related invariant torus, which is impossible because of the assumption of the independence of integrals (1.1).

We introduce angle variables ψ_1 and ψ_2 mod 2π by formulas

$$\psi_i = \frac{\pi}{\tau_i}\int_{a_i}^{s_i}\frac{ds}{\sqrt{\Phi(s)}}, \quad \tau_i = \int_{a_i}^{b_i}\frac{ds}{\sqrt{\Phi(s)}}, \quad s_i \in [a_i b_i] \quad (i = 1, 2) \qquad (1.4)$$

In new variables Eqs. (1.3) assume the form

$$\psi_i^{\cdot} = \frac{\pi}{2\tau_i\,[s_1(\psi_1) - s_2(\psi_2)]} \quad (i = 1, 2) \qquad (1.5)$$

where $s_i(z)$ are real hyperelliptic functions of period 2π which are determined by (1.4). The inverse substitution $(\psi_1, \psi_2) \rightarrow (\varphi_1, \varphi_2)$ reduces Eqs. (1.5) to the form [7]

$$\varphi_i^{\cdot} = \frac{\pi}{2\tau_i\Lambda} \quad (i = 1, 2) \qquad (1.6)$$

$$\Lambda = \frac{1}{2\pi}\left(\int_0^{2\pi} s_1(x)\,dx - \int_0^{2\pi} s_2(x)\,dx\right) \quad (\Lambda \neq 0)$$

These equations determine the conditionally periodic motion on T^2 $\{\varphi_1, \varphi_2 \bmod 2\pi\}$. The ratio of frequencies (the gyration number) is $\gamma = \tau_1/\tau_2$ which evidently depends on I_1 and I_2. That function is not constant at least for small values of parameter μ.

2. The eigenrotation problem. Let us investigate the motion of the body in terms of Euler angles ϑ, φ and ψ. Obviously $x_1, x_2, \ldots, x_6$ are conditionally periodic functions of time. Since $\cos\vartheta = x_6$ and $0 \leqslant \vartheta \leqslant \pi$, function $\vartheta(t)$ is also conditionally periodic.

Lemma 2. If at the initial instant of time $I_1\mu^2 < 4I_2^2$, there exists an $\varepsilon > 0$ such that for all t

$$|x_6(t)| < 1 - \varepsilon \qquad (2.1)$$

Proof. If $|x_6| = 1$, then $x_3 = x_4 = x_5 = 0$. Integrals (1.1) yield the equalities $x_1^2 + x_2^2 = I_1/4$ and $|x_1| = |I_2/\mu|$ $(\mu \neq 0)$. Hence $x_2^2 = I_1/4 - I_2^2/\mu^2$. Consequently, if at some instant of time the equality $|x_6| = 1$ is satisfied at E, then $I_1\mu^2 \geqslant 4I_2^2$. Since set E is compact, inequality (2.1) is valid under conditions of the lemma for some $\varepsilon > 0$.

Note. If $I_1\mu^2 \geqslant 4I_2^2$, the dynamic symmetry axis is vertical for some initial data that satisfy this inequality.

We shall use the following terminology of celestial mechanics [9, 10]. The mean motion of quantity $\xi(t)$ is $\lambda = \text{const}$, if for all t

$$\xi(t) = \lambda t + O(1)$$

If $\xi(t) = \lambda t + o(t)$ when $t \rightarrow \infty$, i.e.

$$\lim_{t\to\infty}\frac{\xi(t) - \lambda t}{t} = 0$$

the principal motion of the quantity $\xi(t)$ is λ.

Statement 1. If conditions of Lemma 2 are satisfied, the spin has an average motion.

Proof. Since $1 - x_6^2 > \varepsilon > 0$ for all t, hence

$$e^{i\varphi} = \frac{x_5 + ix_4}{\sqrt{1 - x_6^2}}$$

is a two-frequency conditionally periodic function of time. By the Bohl theorem about the argument [11]

$$\varphi = (m\omega_1 + n\omega_2)\, t + f\,(t)$$

where m and n are integers and f is a conditionally periodic function of t. Hence $\varphi = \lambda t + O\,(1)$.

If at some instant of time $t = t'$ the equality $x_6^2 = 1$ is satisfied, then angle φ is not formally determined. In such case we proceed as follows. We know that

$$\varphi^{\cdot} = x_3 - x_6 \frac{x_1x_4 + x_2x_5}{1 - x_6^2} = \frac{x_3\,(4 - 3x_6^2)}{4\,(1 - x_6^2)}$$

Using the l'Hospital rule for expanding the indeterminate form for $I_1 \neq 0$, we obtain

$$\lim_{t\to t'} \varphi^{\cdot} = \frac{I_2}{2I_1}, \quad \lim_{t\to t'-0} \varphi\,(t) = \lim_{t\to t'+0} \varphi\,(t) = \varphi'$$

It is reasonable to assume on the basis of the last equality that $\varphi\,(t') = \varphi'$. Function $\varphi\,(t)$ is then determinate and continuous for all $t \in (-\infty,\ \infty)$.

This reasoning shows the expediency of analyzing spin even then, when the axis of symmetry can assume a vertical position.

Theorem 1. Let $I_1 \neq 0$ and $I_1\mu^2 \neq 4I_2^2$. If for given constants of integrals I_1 and I_2 frequencies ω_1 and ω_2 are commensurable, the spin has an average motion. If ω_1 and ω_2 are incommensurable, the spin has a principal motion that depends only on I_1 and I_2.

Proof. Statement 1 implies that it is sufficient to consider the case when $I_1\mu^2 > 4I_2^2$. If the ratio ω_1/ω_2 is rational, then $\varphi^{\cdot}$ is a continuous periodic function of time (at points where $x_6^2 = 1$, function $\varphi^{\cdot}$ is assumed to be equal $I_2/2I_1$). Hence in that case $\varphi = \lambda t + O\,(1)$.

Let us assume now that the ratio ω_1/ω_2 is irrational and consider on T^2 $\{\varphi_1,\ \varphi_2 \bmod 2\pi\}$ the circle $S^1 = \{(\varphi_1,\ \varphi_2) \in T^2 : \varphi_1 = \varphi_1^\circ\}$. On S^1 the variable $\varphi_2 \bmod 2\pi$ is an angle variable. Let us determine on the direct product $S^1 \times [0,\ 2\pi/\omega_1]$ the function

$$F\,(\varphi_2,\ t) = \int_0^t \Phi\,(\omega_1\tau + \varphi_1^\circ,\ \omega_2\tau + \varphi_2)\, d\tau, \quad \varphi_2 \in S^1, \quad t \in \left[0,\ \frac{2\pi}{\omega_1}\right]$$

where $\varphi^{\cdot} = \Phi\,(\varphi_1,\ \varphi_2)$. It is clear that $f\,(\varphi_2) = F\,(\varphi_2,\ 2\pi/\omega_1)$ defines the variation of angle φ during the time taken by a point on T^2 moving along an irrational winding from point $(\varphi_1^\circ,\ \varphi_2) \in S^1$, to return to S^1.

Let us prove that $f\,(\varphi_2)$ is Riemann integrable.

If $f\,(\varphi_2)$ is discontinuous at point $\varphi_2 = \varphi_2'$, the trajectory $(\omega_1 t + \varphi_1^\circ,\ \omega_2 t + \varphi_2')$, $0 \leqslant t \leqslant 2\pi/\omega_1$ passes through points on T^2 where $x_6^2 = 1$. Since there are four such points, $f\,(\varphi_2)$ can have only a finite number of discontinuity points. It is therefore sufficient to prove the boundedness of that function.

Let us prove that $F\,(\varphi_2,\ t)$ is bounded on $S^1 \times [0,\ 2\pi/\omega_1]$.

For this we consider the behavior of angle φ when point $m\,(t) = (\omega_1 t + \varphi_1^\circ,\ \omega_2 t + \varphi_2)$ lies close to points $a_1, \ldots, a_4$, where $x_6^2 = 1$. Since $I_1\mu^2 \neq 4I_2^2$, hence the Jacobian

$$\frac{\partial\,(I_1I_2I_3I_4)}{\partial\,(x_1x_2x_3x_6)}$$

is nonzero at points $a_1, \ldots, a_4 \in T^2$. It is, consequently, possible to take in the small neighborhoods of these points the variables x_4 and x_5 as the local coordinates on T^2, and the variables x_1, x_2, x_3 and x_6 as single-valued analytic functions of x_4 and x_5.

Let us consider the differential equations

$$x_4^{\cdot} = x_3x_5 - x_2x_6, \quad x_5^{\cdot} = x_1x_6 - x_3x_4 \tag{2.2}$$

where x_1, x_2, x_3 and x_6 are replaced by their expressions in terms of x_4 and x_5. Since $I_1 \neq 0$, Eqs. (2.2) do not have singular points close to a_i $(i = 1, \ldots, 4)$.

There exist fairly small neighborhoods U_i of points a_i in which the oscillation of function $F(\varphi_2, t)$ does not exceed 2π when $m(t) \in U_i$.

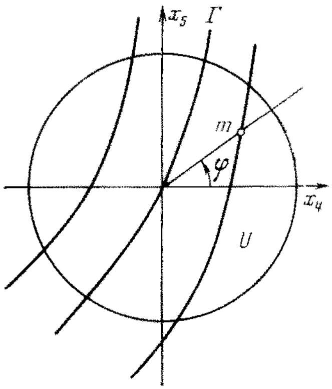

Fig. 1

This is so, since when m moves along the trajectories of Eqs. (2.2), $F(\varphi_2, t)$ coincides with angle φ shown in Fig. 1. The trajectory Γ which passes through point $x_4 = x_5 = 0$ divides $U_i = U$ into two parts in each of which φ varies continuously and has a discontinuity of π-magnitude when passing through Γ. The oscillation of φ is, however, bounded by the number 2π, since in the small neighborhood U the trajectories of Eqs. (2.2) are very nearly straight.

In addition to U_i $(i = 1, \ldots, 4)$ function $1 - x_6^2 > \varepsilon > 0$ and, consequently, function $\Phi(\varphi_1, \varphi_2)$ and the oscillation of function $F(\varphi_2, t)$ are bounded. Summarizing the above, we conclude that $F(\varphi_2, t)$ is bounded in $S^1 \times [0, 2\pi/\omega_1]$.

During the time $t = n2\pi/\omega_1$ angle φ becomes

$$\sum_{k=0}^{n-1} f(k2\pi\omega_2/\omega_1 + \varphi_2) = \sigma_n$$

Since ω_2/ω_1 is irrational, in accordance with Weyl's theorem on uniform distribution we have [12]

$$\lim_{n\to\infty} \frac{\sigma_n}{n} = \lambda$$

The boundedness of function $F(\varphi_2, t)$ implies that

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \frac{2\pi\lambda}{\omega_1} = \Lambda$$

By the same Weyl's theorem the number Λ depends only on I_1 and I_2. The theorem is proved.

The idea of the proof of this theorem stems from Weyl's investigations of the average motion of planet perihelions [10].

3. The problem of motion of the line of nodes. The precession angle is determined by the following formula:

$$\psi^{\cdot} = \frac{x_1x_4 + x_2x_5}{1 - x_6^2} = -\frac{x_3x_6}{4(1 - x_6^2)}$$

If the conditions of Lemma 1 are satisfied, $\psi^{\cdot} = \Psi$ is an analytic function of uniformly varying variables φ_1 and φ_2. In other cases Ψ has a singularity T^2 at points where $x_6^2 = 1$. If $x_6^2 = 1$ at $t = t'$, then by expanding the indeterminacy of the l'Hospital rule, for $I_1 \neq 0$ we obtain

$$\lim_{t\to t'} \Psi^{\cdot}(t) = \mp \frac{I_2}{2I_1} \quad (x_6(t) \to \pm 1)$$

Theorem 2. Let $I_1 \neq 0$ and $I_1\mu^2 \neq 4I_2^2$. If frequencies ω_1 and ω_2 are commensurable, the line of nodes has an average motion. If, however, they are incommensurable, the line of nodes has a principal motion which depends only on I_1 and I_2.

Proof. If the ratio of frequencies ω_1/ω_2 is rational, $\psi^{\cdot}$ is a continuous periodic function of time (at points where $x_6 = \pm 1$ it is assumed to equal $\mp I_2/2I_1$). Hence $\psi = \lambda t + O(1)$.

Let us consider the case of the irrational ratio ω_1/ω_2. If $I_1\mu^2 < 4I_2^2$, then $\Psi(\varphi_1, \varphi_2)$ is continuous on T^2, and the proof of the theorem follows from the theorem on averaging [6]. If, however, $I_1\mu^2 > 4I_2^2$, then, as in the proof of Theorem 1, we introduce the function

$$F(\varphi_2, t) = \int_0^t \Psi^{\cdot}(\omega_1 t + \varphi_1^{\circ}, \ \omega_2 t + \varphi_2)\, dt, \quad \varphi_2 \in S^1, \quad t \in \left[0, \frac{2\pi}{\omega_1}\right]$$

To prove its boundedness we again consider the neighborhoods U_i of points a_i ($i = 1 \ldots 4$). In regions U_i in which x_6 is close to 1 we have the identity

$$\psi^{\cdot} = 2\varphi^{\cdot} + f, \quad f = -\frac{x_3(1-6x_6)}{4(1+x_6)}$$

When $m(t) \in U_i$, the integral of f with respect to time (since f is continuous in U_i) and the integral of $2\varphi^{\cdot}$ are bounded. The motion of other regions where x_6 is close to -1 is similarly analyzed. Outside U_i ($i = 1 \ldots 4$) function Ψ is bounded and, consequently, the oscillation of F is also bounded. Summarizing the above, we conclude that $F(\varphi_2, t)$ is bounded on $S^1\{\varphi_2 \bmod 2\pi\} \times [0, 2\pi/\omega_1]$. To complete the proof it remains to apply Weyl's theorem on uniform distribution.

Statement 2. If $I_1\mu^2 \neq 4I_2^2$, function $\Psi(\varphi_1, \varphi_2)$ is Lebesgue integrable on $T^2\ \{\varphi_1, \varphi_2 \bmod 2\pi\}$.

Proof. If $I_1\mu^2 < 4I_2^2$, then Ψ is continuous on T^2, and the statement is evidently correct. If $I_1\mu^2 > 4I_2^2$, function Ψ is continuous everywhere, except at points $a_1, \ldots, a_4$, where $x_6^2 = 1$. Hence it is sufficient to prove that Ψ is integrable in the small neighborhoods of points a_i ($i = 1, \ldots, 4$). Since $I_1\mu^2 \neq 4I_2^2$, it is possible to take x_4 and x_5 as the local coordinates in U_i. The Jacobian of transformation

$$\frac{\partial(\varphi_1, \varphi_2)}{\partial(x_4, x_5)}$$

is analytic with respect to x_4 and x_5. By the formula of substitution of variables we have

$$\iint_{U_i} \Psi(\varphi_1, \varphi_2)\, d\varphi_1 d\varphi_2 = \iint_{U_i} \Psi(x_4, x_5) \frac{\partial(\varphi_1, \varphi_2)}{\partial(x_4, x_5)}\, dx_4 dx_5$$

We use the equality

$$\Psi = -\frac{x_3 x_6}{4(x_4^2 + x_5^2)}$$

Functions x_3 and x_6 are analytic in U_i with respect to x_4 and x_5, and $x_3 = 0$ when $x_4 = x_5 = 0$. Hence the integrand expressed in terms of x_4 and x_5 is of the form

$$F = f(x_4, x_5) / (x_4^2 + x_5^2)$$

where f is an analytic function in U_i, and $f(0, 0) = 0$. In polar coordinates (r, φ): $x_4 = r\cos\varphi$, $x_5 = r\sin\varphi$

$$\iint\limits_{U_i} F\,dx_4dx_5 = \iint\limits_{U_i} \frac{f}{r}\,dr\,d\varphi$$

Since f/r is continuous and bounded in the deleted neighborhood of points a_i, hence F is Lebesgue integrable in region U_i $(i = 1, \ldots, 4)$. The theorem is proved.

Theorem 3. For small μ

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} \Psi(\varphi_1, \varphi_2)\,d\varphi_1 d\varphi_2 = 0$$

To prove this theorem we need the following lemma.

Lemma 3. Let the contraction of function $f(x_1 \ldots x_6)$ onto the invariant torus T^2 be Lebesgue integrable. Then

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(\varphi_1, \varphi_2)\,d\varphi_1 d\varphi_2 = \oint\limits_{T^2} \frac{f}{V_4}\,d\sigma$$

where V_4 is the four-dimensional volume of the parallelepiped constructed on vectors grad I_i $(i = 1, \ldots, 4)$ as its sides, and $d\sigma$ is area element on T^2 as a surface in R^6 $\{x_1 \ldots x_6\}$.

Proof. In some neighborhood of the invariant torus T^2 $\{\varphi_1, \varphi_2 \bmod 2\pi\}$ in R^6 it is possible to make the invertible substitution of variables

$$x_i = x_i(I_1, \ldots, I_4, \varphi_1, \varphi_2) \qquad (i = 1, \ldots, 6)$$

When $I_3 = 0$ the equations of motion in new variables (I, φ) are of the form

$$I_i^{\cdot} = 0, \ \varphi_j^{\cdot} = \Phi_j(I_1 \ldots I_4); \quad i = 1, \ldots, 4; \ j = 1, 2$$

These equations have an integral invariant of density

$$\rho = M\,\frac{\partial(x_1, \ldots, x_6)}{\partial(I_1, \ldots, I_4, \varphi_1, \varphi_2)}$$

where M is the density of the integral invariant in terms of variables $x_1, \ldots, x_6$. Since $M \equiv 1$ and $\rho = 1$, when $I_3 = 0$, hence in this case

$$\frac{\partial(x_1, \ldots, x_6)}{\partial(I_1, \ldots, I_4, \varphi_1, \varphi_2)} = 1$$

Let us consider vectors

$$\xi_i = \left(\frac{\partial x_1}{\partial I_i}, \ldots, \frac{\partial x_6}{\partial I_i}\right) \quad (i = 1, \ldots, 4)$$

$$\eta_j = \left(\frac{\partial x_1}{\partial \varphi_j}, \ldots, \frac{\partial x_6}{\partial \varphi_j}\right) \qquad (j = 1, 2)$$

Obviously

$$(\text{grad } I_i, \xi_j) = \delta_{ij} \quad (i, j = 1, \ldots, 4)$$

$$(\text{grad } I_i, \eta_k) = 0 \quad (i = 1, \ldots, 4; \ k = 1, 2)$$

where δ_{ij} is the Kronecker delta. We represent vectors ξ_i in the form $\xi_i' + \xi_i''$, where ξ_i' is orthogonal to η_1 and η_2, and ξ_i'' can be expanded in terms of η_1 and η_2. Then

$$V_6(\xi_1 \ldots \xi_4\eta_1\eta_2) = \left|\frac{\partial(x_1 \ldots x_6)}{\partial(I_1 \ldots I_4, \varphi_1, \varphi_2)}\right| = V_4(\xi_i')\,V_2(\eta_j) = 1 \qquad (3.1)$$

where $V_n(a_1 \ldots a_n)$ denotes the n-dimensional volume of the parallelepiped constructed on vectors $a_1, \ldots, a_n$ as its sides. Since again

$$(\text{grad } I_i, \xi_j') = \delta_{ij}$$

hence

$$V_4(\text{grad } I_i)\ V_4(\xi_j') = 1$$

Taking into account (3.1), we obtain

$$V_4(\text{grad } I_i) = V_2(\eta_j)$$

and, since by definition of the area element $d\sigma = V_2(\eta_1, \eta_2)\, d\varphi_1 d\varphi_2$, hence

$$\int_0^{2\pi}\int_0^{2\pi} f d\varphi_1 d\varphi_2 = \int_0^{2\pi}\int_0^{2\pi} \frac{f V_2(\eta_1, \eta_2)}{V_4(\text{grad } I_i)} d\varphi_1 d\varphi_2 = \oint_{T^2} \frac{f}{V_4} d\sigma$$

The lemma is proved.

Proof of Theorem 3. Let us consider the transformation $\pi : R^6 \to R^6$ defined by the formula $y = \pi(x)$, where $x = (x_1 \ldots x_6)$ and $y = (-x_1 - x_2 x_3 x_4 x_5 - x_6)$. The mapping of π, a linear orthogonal transformation, is the product of three mirror images relative to the coordinate hyperplanes. When μ is small, each of the two invariant tori, which constitute the common level of integrals, transforms into itself (see the proof of Lemma 1).

Since $\pi : T^2 \to T^2$ retains its area, the Jacobian of that transformation is equal unity and, consequently,

$$\oint_{T^2} \frac{\Psi(\pi(x))}{V_4(\pi(x))} d\sigma = \oint_{T^2} \frac{\Psi(x)}{V_4(x)} d\sigma \qquad (3.2)$$

By Gramme's formula

$$V_4(\text{grad } I_k) = \sqrt{\det(\text{grad } I_i, \text{ grad } I_j)} \quad (i, j, k = 1, \ldots, 4)$$

With the use of this formula it is possible to prove that $V_4(\pi(x)) = V_4(x)$. Since $\Psi(\pi(x)) = -\Psi(x)$, formula (3.2) yields the equality

$$\int_0^{2\pi}\int_0^{2\pi} \Psi(\varphi_1, \varphi_2)\, d\varphi_1 d\varphi_2 = \oint_{T^2} \frac{\Psi}{V_4} d\sigma = 0$$

The theorem is proved.

Corollary. If μ is small and the ratio of frequencies ω_1/ω_2 irrational, the principal motion of the line of nodes is zero, since by the theorem on uniform distribution [6, 12]

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = \lambda = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \Psi(\varphi_1, \varphi_2)\, d\varphi_1 d\varphi_2 = 0$$

The author thanks V. V. Rumiantsev and Iu. A. Arkhangel'skii for their interest in this work.

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Translated by J. J. D.